Dispersion relations and wave operators in self-similar quasicontinuous linear chains

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We construct self-similar functions and linear operators to deduce a self-similar variant of the Laplacian operator and of the D'Alembertian wave operator. The exigence of self-similarity as a symmetry property requires the introduction of nonlocal particle-particle interactions. We derive a self-similar linear wave operator describing the dynamics of a quasicontinuous linear chain of infinite length with a spatially self-similar distribution of nonlocal interparticle springs. The self-similarity of the nonlocal harmonic particle-particle interactions results in a dispersion relation of the form of a Weierstrass-Mandelbrot function that exhibits self-similar and fractal features. We also derive a continuum approximation, which relates the self-similar Laplacian to fractional integrals, and yields in the low-frequency regime a power-law frequency-dependence of the oscillator density.

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I. INTRODUCTION

The development of fractal geometry by Mandelbrot [1] already launched a scientific revolution in the 1970s whereas the mathematical roots originate much earlier in the 19th century [2]. However, it is only recently that the problems of fractal and self-similar media have become a subject in analytical mechanics. This is true in statics and dynamics. One important reason for this seems to be the considerable mathematical difficulty even in defining physical problems on fractals and this is even more so for the construction of analytical solutions to these problems. Inspired by the exotic electromagnetic properties that fractal gaskets reveal when used as "fractal antennae" [3,4], it had also been found that fractal gaskets exhibit exotic vibrational properties [5] that may open the door for new technological applications. An improved understanding of these properties could raise an enormous new interdisciplinary field for basic research and applications in a wide range of mechanical disciplines including fluid mechanics and the mechanics of granular media and solids. However a "fractal mechanics" has yet to be developed. Some crucial steps have already been performed (see papers [5–12] and the references therein). In [6] the fractal counterpart of the static harmonic calculus has been described by means of the Sierpinski gasket by employing a graph theoretical approach to define the Laplacian on the Sierpinski gasket. In paper [5] the vibrational spectrum of a Sierpinski gasket was numerically modeled; however, no rigorous approach was given. A significant contribution by analyzing Fourier spectra of fractal Sierpinski signals has been given in [9]. Closed form solutions for the dynamic Green's function and the vibrational spectrum of a linear chain with spatially exponential properties are given in a recent paper [11].

Several key contributions of fractal chains and lattices have been presented in the literature [13–15]. In Ref. [13] the effect of scaled connectivity on coupled lattices has been analyzed. An experimental study of the existence of localized excitations in fractal (antenna) supermolecular structures has been presented in Ref. [14]. Solutions of the Schrödinger equation on several fractal lattices (quantum chains) have been studied in Ref. [15] by employing a recursive technique to determine the quantum-mechanical Green's function of the generator lattices. The lattices considered in [15] have self-similar features but no translational invariance. All these models [13–15] address problems on *discrete* lattices with fractal features. A similar fractal type of linear chain as in the present paper has been considered very recently by Tarasov [7]. Unlike in the present paper the chain considered in Ref. [7] is *discrete*, i.e., there remains a characteristic length scale that is given by the next-neighbor distance of the particles.

In contrast to all these works we analyze in the present paper vibrational properties in a *quasicontinuous* linear chain with (in the self-similar limiting case) infinitesimal lattice spacing with a nonlocal spatially self-similar distribution of power-law-scaled harmonic interparticle interactions (springs). In this way there is no characteristic length scale in our chain.

In the present paper we utilize elements of lattice dynamics of linear chains together with a methodology to account for self-similarity. The demonstration is organized as follows: Sec. II is devoted to the construction of self-similar functions and operators. By using this approach we construct a self-similar analog to the Laplace operator to define a self-similar variant of the wave equation for a self-similar linear dynamic system in Sec. III. We hope the present approach launches some interdisciplinary work and collaborations also in fields concerned with fractal aspects of turbulence and fluid mechanics. It seems there are analog situations [16] where the present approach could be useful.

Some more detailed remarks to the chain considered by Tarasov [7] are as follows. There is a crucial difference between the discrete Tarasov chain and the quasicontinuous chain being subject of our paper: the Tarasov chain is discrete, i.e., there is a well defined distance between next-

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neighbor particles. In the Tarasov chain each particle interacts with particles of order N^s, where $N>1$ is an integer and $s=0,1,2,\ldots$ assumes all positive integers including $s=0$, which corresponds to the next neighbor. The Tarasov nonlocal harmonic interaction exhibits fractal but not self-similar features.

In contrast we consider here a *quasicontinuous* chain with harmonic *exact self-similar* nonlocal interparticle interactions. In our chain any particle at space point x interacts harmonically (spring constants ξ^s, $0<\xi<1$) with particles located at $x\pm hN^s$, where $N\in \mathrm{R}(N>1)$ can also be noninteger and $s=-\infty,\ldots,0,\ldots+\infty$ is running over all positive and negative integers including zero.[1] In contrast to the Tarasov chain, the elastic energy (density) introduced in our chain is an exact self-similar function.

II. CONSTRUCTION OF SELF-SIMILAR FUNCTIONS AND LINEAR OPERATORS

In this paragraph we define the term "self-similarity" with respect to functions and operators. We call a scalar function $\phi(h)$ *exact self-similar* with respect to variable h if the condition,

$$\phi(Nh)=\Lambda\phi(h), \tag{1}$$

is satisfied for all values $h>0$ of the scalar variable h. We call Eq. (1) the "affine problem[2]" where N is a fixed parameter and $\Lambda=N^{\delta}$ represents a continuous set of admissible eigenvalues. The band of admissible $\delta=\frac{\ln\Lambda}{\ln N}$ is to be determined. A function $\phi(h)$ satisfying Eq. (1) for a certain N and admissible $\Lambda=N^{\delta}$ represents an unknown "solution" to the affine problem of the form $\phi_{N,\delta}(h)$, and is to be determined.

As we will see below for a given N solutions $\phi(h)$ *exist* only in a certain range of admissible Λ. From the definition of the problem, it follows that if $\phi(h)$ is a solution of Eq. (1) it is also a solution of $\phi(N^s h)=\Lambda^s\phi(h)$, where $s\in \mathrm{Z}$ is discrete and can take all positive and negative integers including zero. We emphasize that noninteger s are not admitted. The discrete set of pairs Λ^s, N^s are for all $s\in\mathrm{Z}$ related by a power law with the same power δ, i.e., $\Lambda=N^{\delta}$; hence $\Lambda^s=(N^s)^{\delta}$. By replacing Λ and N by Λ^{-1} and N^{-1} in Eq. (1), it defines the identical problem. Hence we can restrict our considerations on fixed values of $N>1$.

We can consider affine problem (1) as the eigenvalue problem for a linear operator $\hat{A}_N$ with a certain given fixed parameter N and eigenfunctions $\phi(h)$ to be determined, which correspond to an *admissible* range of eigenvalues $\Lambda=N^{\delta}$ (or equivalently to an admissible range of exponent $\delta=\ln\Lambda/\ln N$). For a function $f(x,h)$ we denote by $\hat{A}_N(h)f(x,h)=:f(x,Nh)$ when the affine transformation is only performed with respect to variable h.

We assume $\Lambda,N\in\mathrm{R}$ for physical reasons without too much loss of generality to be real and positive. Moreover, definition (1) does not necessarily need to be restricted to the scalar case. We also can define self-similarity of a vector valued function $\vec{\phi}(\vec{h})\in\mathrm{R}^n$ in the fully analogous manner, where $\mathbf{N}$ and $\mathbf{\Lambda}$ are Hermitian positive-definite $n\times n$ matrices. In this paper however we confine us to the scalar case. For our convenience we define the "affine" operator $\hat{A}_N$ as follows

$$\hat{A}_N f(h)=:f(Nh). \tag{2}$$

It is easily verified that the affine operator $\hat{A}_N$ is *linear*, that is, it fulfills the relation

$$\hat{A}_N[c_1f_1(h)+c_2f_2(h)]=c_1f_1(Nh)+c_2f_2(Nh), \tag{3}$$

and

$$\hat{A}_N^s f(h)=f(N^s h),\quad s=0\pm1,\pm2,\ldots,\pm\infty. \tag{4}$$

From this follows that we can define affine operator functions for any smooth function $g(\tau)$ that can be expanded into a Maclaurin series as

$$g(\tau)=\sum_{s=0}^{\infty}a_s\tau^s. \tag{5}$$

We define an affine operator function in the form

$$g(\xi\hat{A}_N)=\sum_{s=0}^{\infty}a_s\xi^s\hat{A}_N^s, \tag{6}$$

where ξ denotes a scalar parameter. The operator function that is defined by Eq. (6) acts on a function $f(h)$ as follows

$$g(\xi\hat{A}_N)f(h)=\sum_{s=0}^{\infty}a_s\xi^s f(N^s h), \tag{7}$$

where relation (4) with expansion (6) has been used. The convergence of series (7) has to be verified for a function $f(h)$ to be admissible. An explicit representation of the affine operator $\hat{A}_N$ can be obtained when we write $f(h)=f(e^{\ln h})=\bar{f}(\ln h)$. Hence application of $\hat{A}_N$ on $f(h)$ is nothing but a translation in the variable v in $\bar{f}(v=\ln h)$. We introduce

$$\hat{A}_N\bar{f}(v)=\bar{f}(v+\ln N),\quad \bar{f}(v)=f(e^v), \tag{8}$$

such that

$$\hat{A}_N f(h)=f(Nh)=\bar{f}(\ln N+\ln h)=e^{\ln N(d/dv)}f(e^v)\big|_{v=\ln h}, \tag{9}$$

where we assume that $f(h)$ is a sufficiently smooth function. The exponential operator $e^{\ln N(d/dv)}$ performs a translation in the variable v by $\ln N$ and is defined by

$$e^{\tau(d/dv)}=\sum_{s=0}^{\infty}\frac{\tau^s}{s!}\frac{d^s}{dv^s}\quad\text{with}\quad e^{\tau(d/dv)}\bar{f}(v)\big|_{v=v_0}=\bar{f}(v_0+\tau). \tag{10}$$

Hence the affine operator $\hat{A}_N$ can be written explicitly in the form

$$\hat{A}_N(h)=e^{\ln N\{d/[d(\ln h)]\}}. \tag{11}$$

[1] Owing to this symmetry in s we confine to $N>1$ without any loss of generality.

[2] We restrict here to affine transformations $h'=Nh+c$ with $c=0$.

This relation is immediately verified in view of

$$\hat{A}_N(h)f(h) = e^{\ln N\{d/[d(\ln h)]\}}f(e^{\ln h}) = f(e^{\ln h+\ln N}) = f(Nh). \tag{12}$$

With this machinery we are now able to construct self-similar functions and operators. This will be performed in the next subsection in order to define the wave propagation problem for a self-similar quasicontinuous linear chain (Sec. IV).

A. Construction of self-similar functions

A self-similar function solving problem (1) is formally given by the series

$$\phi(h) = \sum_{s=-\infty}^{\infty} \Lambda^{-s}\hat{A}_N^s f(h) = \sum_{s=-\infty}^{\infty} \Lambda^{-s} f(N^s h) \tag{13}$$

for any function $f(h)$ for which series (13) is uniformly convergent for all h. We introduce the self-similar operator

$$\hat{T}_N = \sum_{s=-\infty}^{\infty} \Lambda^{-s}\hat{A}_N^s, \tag{14}$$

which fulfills formally the condition of self-similarity $\hat{A}_N\hat{T}_N=\Lambda\hat{T}_N$ and hence Eq. (13) solves affine problem (1). In view of the symmetry with respect to inversion of the sign of s in Eqs. (13) and (14), we can restrict ourselves to $N>1(N,\Lambda\in\mathbb{R})$ without any loss of generality[3]: let us look for admissible functions $f(t)$ for which Eq. (13) is convergent. To this end we have to demand simultaneous convergence of the partial sums over positive and negative s. Let us assume that (where we can confine ourselves to $t>0$)

$$\lim_{t\to 0} f(t) = a_0 t^{\alpha}. \tag{15}$$

For $t\to\infty$ we have to demand that $|f(t)|$ increases not stronger than a power of t, i.e.,

$$\lim_{t\to\infty} f(t) = c_\infty t^{\beta}, \tag{16}$$

with a_0, c_∞ denoting constants. Both exponents $\alpha,\beta\in\mathbb{R}$ are allowed to take positive or negative values, which do not need to be integers. A brief consideration of partial sums yields the following requirements for $\Lambda=N^{\delta}$: namely, summation over $s<0$ in Eq. (13) requires absolute convergence of a geometrical series leading to the condition for its argument $\Lambda N^{-\alpha}<1$. That is, we have to demand $\delta<\alpha$. The partial sum over $s>0$ requires absolute convergence of a geometrical series leading to the condition for its argument $\Lambda^{-1}N^{\beta}<1$, which corresponds to $\delta>\beta$. Both conditions are simultaneously met if

$$N^{\beta} < \Lambda = N^{\delta} < N^{\alpha}, \tag{17}$$

or equivalently

$$\beta < \delta = \frac{\ln\Lambda}{\ln N} < \alpha. \tag{18}$$

[3]We also can exclude the trivial case $N=1$.

Relations (17) and (18) require additionally $\beta<\alpha$, that is, only functions $f(t)$ with behaviors (15) and (16) with $\beta<\alpha$ are *admissible* in Eq. (13). These cases include certain bounded functions $|f(t)|<M$ corresponding to $\beta=0$. For instance some periodic functions refer to this category.

B. Self-similar analog to the Laplace operator

In the spirit of Eqs. (13) and (14) we construct an exactly self-similar function from the second difference according to

$$\phi(x,h) = \hat{T}_N(h)[u(x+h)+u(x-h)-2u(x)], \tag{19}$$

where $u(\dots)$ denotes an arbitrary smooth continuous field variable and $\hat{T}_N(h)$ expresses that the affine operator $\hat{A}_N(h)$ acts only on the dependence on h, that is, $\hat{A}_N(h)v(x,h)=v(x,Nh)$. We have with $\xi=\Lambda^{-1}$ the expression

$$\phi(x,h) = \sum_{s=-\infty}^{\infty} \xi^s\{u(x+N^s h)+u(x-N^s h)-2u(x)\}, \tag{20}$$

which is a self-similar function with respect to its dependence on h with $\hat{A}_N(h)\phi(x,h)=\phi(x,Nh)=\xi^{-1}\phi(x,h)$ but a regular function with respect to x. The function $\phi(x,h)$ exists if series (20) is convergent. Let us assume that $u(x)$ is a smooth function with a convergent Taylor series for any h. Then we have with $u(x\pm h)=e^{\pm h\frac{d}{dx}}u(x)$ and $u(x+h)+u(x-h)-2u(x)=(e^{h\frac{d}{dx}}+e^{-h\frac{d}{dx}}-2)u(x)$, which can be written as

$$\begin{aligned} u(x+h)+u(x-h)-2u(x) &= 4\sinh^2\left(\frac{h}{2}\frac{d}{dx}\right)u(x) \\ &= h^2\frac{d^2}{dx^2}u(x) + \text{orders } h^{\geq 4}, \end{aligned} \tag{21}$$

thus $\alpha=2$ in criteria (15) is met. If we demand $u(x)$ to be Fourier transformable we should have, as a necessary condition,

$$\int_{-\infty}^{\infty} |u(x)|dx < \infty \tag{22}$$

exist. This is true if $|u(t)|$ tends to zero as $t\to\pm\infty$, as $|t|^{\beta}$ where $\beta<-1$. We have then the condition that

$$\beta < 0 < \delta = -\frac{\ln\xi}{\ln N} < \alpha = 2. \tag{23}$$

We will see below that only $\delta>0$ is *physically admissible*, i.e., compatible with harmonic particle-particle interactions that decrease with increasing particle-particle distance.

The one-dimensional (1D) Laplacian Δ_1 is defined by

$$\Delta_1 u(x) = \frac{d^2}{dx^2}u(x) = \lim_{\tau\to 0}\frac{[u(x+\tau)+u(x-\tau)-2u(x)]}{\tau^2}. \tag{24}$$

Let us now define a self-similar analog to the 1D Laplacian. We emphasize that also other definitions could be imagined.

However, the definition to follow has a certain "physical" justification, as we will see in Sec. III. In analogy to Eq. (24) we put with $\xi = N^{-\delta}$

$$\Delta_{(\delta,N,\tau)}u(x) =: \text{const} \lim_{\tau\to 0} \tau^{-\lambda}\phi(x,\tau) \tag{25}$$

$$= \text{const} \lim_{\tau\to 0} \tau^{-\lambda}\sum_{s=-\infty}^{\infty}\xi^{s}[u(x+N^{s}\tau)+u(x-N^{s}\tau) - 2u(x)], \tag{26}$$

where we have introduced a renormalization multiplier $\tau^{-\lambda}$ with the power λ to be determined to guarantee the limiting case being finite. The constant factor const indicates that there is a certain arbitrariness in this definition and will be chosen conveniently. Let us consider the limit $\tau\to 0$ by the special sequence $\tau_n = N^{-n}h$ with $n\to\infty$ and h being constant. Unlike in 1D case (24), the result of this limiting process depends crucially on the choice of the sequence τ_n. Then we have [by putting in Eq. (25) const$=h^{\lambda}$]

$$\Delta_{(\delta,N,h)}u(x) = \lim_{n\to\infty} N^{\lambda n}\xi^{n}\sum_{s=-\infty}^{\infty}\xi^{s-n}[u(x+N^{s-n}h)+u(x-N^{s-n}h) - 2u(x)], \tag{27}$$

which assumes, by replacing $s-n\to s$, the form

$$\Delta_{(\delta,N,h)}u(x) = \phi(x,h)\lim_{n\to\infty} N^{-(\delta-\lambda)n}, \tag{28}$$

which is only finite and nonzero if $\lambda=\delta$. The "Laplacian" can then be defined simply by

$$\Delta_{(\delta,N,h)}u(x) =: \lim_{n\to\infty} N^{\delta n}\phi(x,N^{-n}h) = \phi(x,h), \tag{29}$$

or by using Eqs. (19) and (21) we can simply write[4]

$$\Delta_{(\delta,N,h)} = 4\hat{T}_N(h)\sinh^2\left(\frac{h}{2}\frac{\partial}{\partial x}\right) = 4\sum_{s=-\infty}^{\infty} N^{-\delta s}\sinh^2\left(\frac{N^{s}h}{2}\frac{\partial}{\partial x}\right), \tag{30}$$

where $\hat{T}_N(h)$ is the self-similar operator defined in Eq. (14). The self-similar analog of Laplace operator defined by Eq. (30) depends on the parameters δ, N, h. We furthermore observe the self-similarity of Laplacian (30), namely,

$$\Delta_{(\delta,N,Nh)} = N^{\delta}\Delta_{(\delta,N,h)}. \tag{31}$$

C. Continuum approximation—link to fractional integrals

For numerical evaluations it may be convenient to utilize a continuum approximation of self-similar Laplacian (30). To this end we put $N=1+\epsilon$ (with $0<\epsilon\ll 1$ thus $\epsilon\approx \ln N$), where ϵ is assumed to be "small" and $s\epsilon = v$ such that $dv\approx\epsilon$ and $N^{s}=(1+\epsilon)^{\frac{v}{\epsilon}}\approx e^{v}$. In this approximation $N^{s}\approx e^{v}$ becomes a (quasi)continuous variable when s runs through $s\in\mathbb{Z}$. Then we can write Eq. (13) in the form

$$\phi(h) = \sum_{s=-\infty}^{\infty} N^{-s\delta}f(N^{s}h) \approx \frac{1}{\epsilon}\int_{-\infty}^{\infty} e^{-\delta v}f(he^{v})dv, \tag{32}$$

which can be further written with $he^{v}=\tau$ $(h>0)$ and $\frac{d\tau}{\tau}=dv$, and $\tau(v\to-\infty)=0$ and $\tau(v\to\infty)=\infty$ as

$$\phi(h) \approx \frac{h^{\delta}}{\epsilon}\int_{0}^{\infty}\frac{f(\tau)}{\tau^{1+\delta}}d\tau. \tag{33}$$

In this continuous approximation the function $\phi(h)$ obeys the same scaling behavior as Eq. (13), namely, $\phi(h\lambda)=\lambda^{\delta}\phi(h)$ but in contrast to Eq. (13) λ can assume any continuous positive value. This is due to the fact that Eq. (33) is holding for $N=1+\epsilon$ with sufficiently small $\epsilon>0$ since in this limiting case there exists for any continuous value $\lambda>0$ an $m\in\mathbb{Z}$ such that $N^{m}\approx\lambda$. The existence requirement for integral (33) leads to the same requirements for $f(t)$ as in Eq. (13), namely, inequality (18). Application of approximate relation (33) to Laplacian (30) yields

$$\Delta_{(\delta,\epsilon,h)}u(x) \approx \frac{h^{\delta}}{\epsilon}\int_{0}^{\infty}\frac{[u(x-\tau)+u(x+\tau)-2u(x)]}{\tau^{1+\delta}}d\tau, \tag{34}$$

where this integral exists for $\beta<0<\delta<2$ and $\beta<-1$ because of the required existence of integral (22) and relation (21). For $0<\delta<1$ we can split Eq. (34) into the two integrals

$$\Delta_{(\delta,\epsilon,h)}u(x) \approx \frac{h^{\delta}}{\epsilon}\int_{0}^{\infty}\frac{[u(x+\tau)-u(x)]}{\tau^{1+\delta}}d\tau + \frac{h^{\delta}}{\epsilon}\int_{0}^{\infty}\frac{[u(x-\tau)-u(x)]}{\tau^{1+\delta}}d\tau. \tag{35}$$

By performing two partial integrations and by taking into account the vanishing boundary terms at $\tau=0$ and $\tau=\infty$ for $0<\delta<2$, we can rewrite Eq. (34) in the form

$$\Delta_{(\delta,\epsilon,h)}u(x) \approx \frac{h^{\delta}}{\delta(\delta-1)\epsilon}\int_{x}^{\infty}(\tau-x)^{1-\delta}\frac{d^2u}{d\tau^2}(\tau)d\tau + \frac{h^{\delta}}{\delta(\delta-1)\epsilon}\int_{-\infty}^{x}(x-\tau)^{1-\delta}\frac{d^2u}{d\tau^2}(\tau)d\tau. \tag{36}$$

We observe here the remarkable fact that this integral is a convolution of the conventional 1D Laplacian $\frac{d^2u}{dx^2}(x)$, namely,

$$\Delta_{(\delta,\epsilon,h)}u(x) \approx \int_{-\infty}^{\infty} g(|x-\tau|)\frac{d^2u}{d\tau^2}(\tau)d\tau, \tag{37}$$

with the kernel

$$g(|x|) = \frac{h^{\delta}}{\delta(\delta-1)\epsilon}|x|^{1-\delta}, \quad \delta\neq 1 \tag{38}$$

where $0<\delta<2$ and $g(|x|)=-\frac{h}{\epsilon}\ln|x|$ for $\delta=1$.

[4] We have to replace $\frac{d}{dx}\to\frac{\partial}{\partial x}$ if the Laplacian acts on a field $u(x,t)$ as in Sec. III.

Further illuminating is the possibility to express Eq. (36) in terms of *fractional integrals*. To this end we put $D=2-\delta>0$, which is positive in the admissible range of δ. For $0<\delta<1$ the quantity D can be identified with the estimated fractal dimension of the fractal dispersion relation of the Laplacian [17], which is deduced in the next section. The Riemann-Liouville fractional integral is defined by (e.g., [18,19])

$$\mathcal{D}_{a,x}^{-D}v(x)=\frac{1}{\Gamma(D)}\int_a^x (x-\tau)^{D-1}v(\tau)d\tau, \tag{39}$$

where $\Gamma(D)$ denotes the Γ function, which represents the generalization of the factorial function to noninteger $D>0$. The Γ function is defined as

$$\Gamma(D)=\int_0^\infty \tau^{D-1}e^{-\tau}d\tau, \quad D>0. \tag{40}$$

For positive integers $D>0$ the Γ function reproduces the factorial function $\Gamma(D)=(D-1)!$ with $D=1,2,\ldots,\infty$. Laplacian (36) can then be expressed in the form for $D\neq 1$ as

$$\Delta_{(\delta=2-D,\epsilon,h)}u(x)\approx\frac{h^{2-D}}{\epsilon}\frac{\Gamma(D)}{(D-1)(D-2)} \times[\mathcal{D}_{-\infty,x}^{-D}+(-1)^D\mathcal{D}_{\infty,x}^{-D}]\Delta_1 u(x), \tag{41}$$

where $\Delta_1 u(x)=\frac{d^2}{dx^2}u(x)$ denotes the conventional 1D Laplacian. In the integral associated with second term of Eq. (41), we have to choose because of $x-\tau<0$ the phase of $-1=e^{\pm i\pi}$ in $(-1)^D$ such that $(x-\tau)^{D-1}(-1)^{D-1}=|x-\tau|^{D-1}\in \mathrm{R}$ remains real, e.g., for instance by putting simultaneously $x-\tau=e^{-i\pi}|x-\tau|$ and $-1=e^{i\pi}$.

The present continuum approximation holds mathematically for $0<\delta<2$ with $\beta<-1$ and it will be demonstrated in the next section that indeed we have to demand for any physical system $\delta>0$ in Laplacian (30). This is due to the fact that physical interparticle interactions have to decay with increasing interparticle distance and to diverge when the interparticle distance tends to zero. Hence the requirement of convergence of the above integrals together with the demand for the Laplacian to describe a physical system with harmonically interacting particles restrict δ within the interval

$$0<\delta<2. \tag{42}$$

In the next section it will be outlined that $0<\delta<1$ is the range where the dispersion relation of the Laplacian reveals fractal features.

III. PHYSICAL MODEL

We consider an infinitely long quasicontinuous linear chain of identical particles. Any space point x corresponds to a "material point" or particle. The mass density of particles is assumed to be spatially homogeneous and equal to one for any space point x. Any particle is associated with one degree of freedom, which is represented by the displacement field $u(x,t)$, where x is its spatial (Lagrangian) coordinate and t indicates time. In this sense we consider a quasicontinuous spatial distribution of particles. Any particle at space point x is nonlocally connected by harmonic springs of strength ξ^s to particles located at $x\pm N^s h$, where $N>1$ and $N\in \mathrm{R}$ is not necessarily integer, $h>0$, and $s=0,\pm 1,\pm 2,\ldots,\pm\infty$. The requirement of decreasing spring constants with increasing particle-particle distance leads to the requirement that $\xi=N^{-\delta}<1(N>1)$, i.e., only chains with $\delta>0$ are physically admissible. In order to get exact self-similarity we avoid the notion of "next-neighbor particles" in our chain, which would be equivalent to the introduction of an internal length scale (the next-neighbor distance). To admit particle interactions over arbitrarily close distances $N^s h\to 0$ ($s\to-\infty$, h=const), our chain has to be *quasicontinuous*. This is the principal difference to the *discrete* chain considered recently by Tarasov [7], which is not self-similar.

The Hamiltonian that describes our chain can be written as

$$H=\frac{1}{2}\int_{-\infty}^{\infty}[\dot{u}^2(x,t)+\mathcal{V}(x,t,h)]dx. \tag{43}$$

In the spirit of Eq. (13) the elastic energy density $\mathcal{V}(x,t,h)$ is assumed to have been constructed self-similarly, namely,[5]

$$\mathcal{V}(x,t,h)=\frac{1}{2}\hat{T}_N(h)\{[u(x,t)-u(x+h,t)]^2 +[u(x,t)-u(x-h,t)]^2\}, \tag{44}$$

where $\hat{T}_N(h)$ is self-similar operator (14) with $\xi=\Lambda^{-1}=N^{-\delta}$ to arrive at

$$\mathcal{V}(x,t,h)=\frac{1}{2}\sum_{s=-\infty}^{\infty}\xi^s\{[u(x,t)-u(x+hN^s,t)]^2 +[u(x,t)-u(x-hN^s,t)]^2\}. \tag{45}$$

The elastic energy density $\mathcal{V}(x,t,h)$ fulfills the condition of self-similarity with respect to h, namely,

$$\hat{A}_N(h)\mathcal{V}(x,t,h)=\mathcal{V}(x,t,Nh)=\xi^{-1}\mathcal{V}(x,t,h). \tag{46}$$

The criteria of convergence of Eq. (45) yields $\alpha=2$ as for Laplacian (20). To determine β we have to demand that $u(x,t)$ be Fourier transformable[6]; thus we have to have an asymptotic behavior of $|u(x\pm\tau,t)|\to 0$ as τ^β, where $\beta<-1$ as $\tau\to\infty$. From this follows $|u(x,t)-u(x\pm\tau,t)|^2$ behaving then as $|u(x,t)|^2$. Hence, elastic energy density (45) converges if

$$2\beta<0<\delta<\alpha=2, \tag{47}$$

where $\beta<-1$. However the requirement of the convergence of the equation of motion [Eq. (51) below] depends on the behavior of $|u(x,t)-u(x\pm\tau,t)|$ for $\tau\to\infty$. From this follows that

[5] The additional factor of 1/2 in the elastic energy avoids double counting.

[6] This assumption defines the (function) space of eigenmodes and corresponds to infinite body boundary conditions.

$$\beta < 0 < \delta < \alpha = 2. \tag{48}$$

Relation (48) determines the range of the admissible values of δ in order to achieve convergence. We emphasize that physically only chains are admissible with $\delta > 0$ in order to have decreasing interparticle spring constants with increasing interparticle distance. We will see below that the requirement $\delta > 0$ works out in a natural way as a consequence of the convergence requirement of the dispersion relation.

If Eq. (48) is fulfilled the convergence of the equation of motion [Eqs. (50) and (51) below] is guaranteed since relation (47) is also fulfilled (since $\beta < -1$).

The equation of motion is obtained by

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\delta H}{\delta u}, \tag{49}$$

(where $\delta / \delta u$ stands for a functional derivative) to arrive at

$$\frac{\partial^2 u}{\partial t^2} = -\sum_{s=-\infty}^{\infty} \xi^s [2u(x,t) - u(x + hN^s, t) - u(x - hN^s, t)], \tag{50}$$

$$\frac{\partial^2 u}{\partial t^2} = \Delta_{(\delta,N,h)} u(x,t), \tag{51}$$

with the self-similar Laplacian $\Delta_{(\delta,N,h)}$ of Eq. (30). We can rewrite Eq. (51) in the compact form of a wave equation

$$\Box_{(\delta,N,h)} u(x,t) = 0, \tag{52}$$

where $\Box_{(\delta,N,h)}$ is the self-similar analogue of the D'Alembertian wave operator having the form

$$\Box_{(\delta,N,h)} = \Delta_{(\delta,N,h)} - \frac{\partial^2}{\partial t^2}. \tag{53}$$

D'Alembertian (53) with Laplacian (30) describes the wave propagation in the self-similar chain with Hamiltonian (43). It appears to be useful and feasible to extend this approach to a general description of wave propagation phenomena in fractal and self-similar material systems.

Now the goal is to determine the dispersion relation, which is constituted by the (negative) eigenvalues of (semi-)negative definite Laplacian (30). To this end we make use of the fact that the displacement field $u(x,t)$ is Fourier transformable [guaranteed by choosing $\beta < -1$ in Eq. (48)] and that the exponentials e^{ikx} are eigenfunctions of self-similar Laplacian (30). We hence write the Fourier integral,

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(k,t) e^{ikx} dk, \tag{54}$$

to rewrite Eq. (51) for the Fourier amplitudes $\tilde{u}(k,t)$ in the form

$$\frac{\partial^2 \tilde{u}}{\partial t^2}(k,t) = -\bar{\omega}^2(k) \tilde{u}(k,t). \tag{55}$$

In this equation $-\bar{\omega}^2(k)$ is obtained by replacing $\frac{\partial}{\partial x}$ by ik in Eq. (30) to arrive at

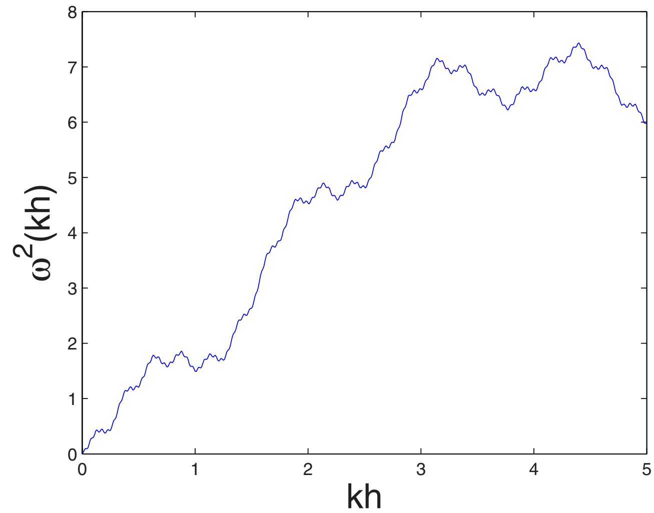

FIG. 1. (Color online) Dispersion relation $\omega^2(kh)$ in arbitrary units for $\delta = \log 4 / \log 5$ of fractal dimension $D \approx 1.14$ and $N=5$.

$$\bar{\omega}^2(k) = \omega^2(kh) = 4\hat{T}_N(h) \sin^2\left(\frac{kh}{2}\right), \tag{56}$$

which yields, by applying the self-similar operator $\hat{T}_N(h)$ [Eqs. (13) and (14)],

$$\omega^2(kh) = 4 \sum_{s=-\infty}^{\infty} N^{-\delta s} \sin^2\left(\frac{khN^s}{2}\right). \tag{57}$$

Equation (57) describes a *Weierstrass-Mandelbrot function*, which is a continuous but for $0 < \delta \leq 1$ nowhere differentiable function [1] and fulfills the condition of self-similar symmetry, namely,

$$\omega^2(Nkh) = N^{\delta} \omega^2(kh). \tag{58}$$

A similar consideration as the above shows that convergence of Eq. (57) restricts δ to the range

$$0 < \delta < 2. \tag{59}$$

Hence *only* exponents δ in interval (59) are *admissible* in Hamiltonian (43) with elastic energy density (45) in order to have a "well-posed" problem. Condition (59) includes automatically "physical admissibility," which requires $\delta > 0 (\xi = N^{-\delta} < 1)$ in Eq. (45) in order to have spring constants $N^{-\delta s} = (L_s / h)^{-\delta}$ that decrease monotonously, and tend to zero with increasing interparticle distances $L_s \to \infty$ and diverge for interparticle distances $L_s \to 0$.

It was shown by Hardy [17] that for $\xi N > 1$ and $\xi = N^{-\delta} < 1$ or equivalently for

$$0 < \delta < 1, \tag{60}$$

the Weierstrass-Mandelbrot function of form (57) is not only self-similar but also a *fractal* curve of (estimated) noninteger fractal (Hausdorff) dimension $D = 2 - \delta > 1$. Figures 1–3 show dispersion curves $\omega^2(kh)$ for different decreasing values of admissible $0 < \delta < 1$ and increasing fractal dimension D. The increase in the fractal dimension from Figs. 1–3 is

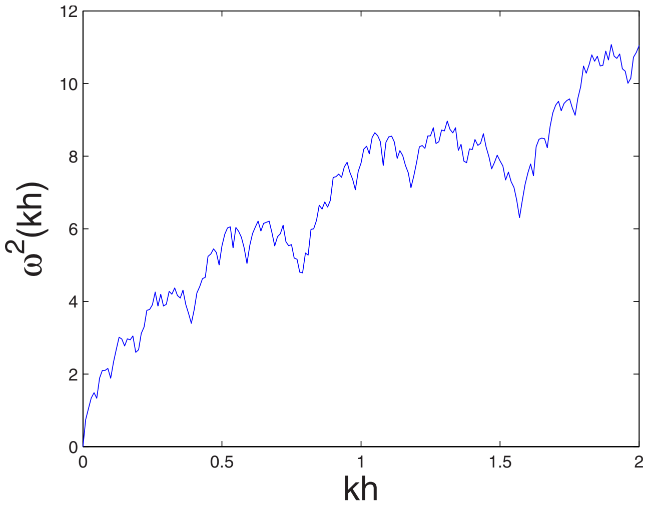

FIG. 2. (Color online) Dispersion relation $\omega^2(kh)$ in arbitrary units for $\delta=0.5$ of fractal dimension $D=1.5$ and $N=2$.

indicated by the increasingly irregular harsh behavior of the curves.

To evaluate Eq. (57) approximately it is convenient to replace the series by an integral utilizing a similar substitution as in Sec. II C ($\epsilon \approx \ln N$). By doing so we smoothen Weierstrass-Mandelbrot function (57). It is important to notice that the resulting approximate dispersion relation is hence differentiable and has no more fractal dimension $D>1$ in interval (60). For sufficiently "small" $|k|h(h>0)$, i.e., in the long-wave regime we arrive at

$$\omega^2(kh) \approx \frac{(h|k|)^\delta}{\epsilon} C, \tag{61}$$

which is only finite if $(|k|h)^\delta$ is in the order of magnitude of ϵ or smaller. This regime, which includes the limit $k\to 0$, is hence characterized by a power-law behavior $\bar{\omega}(k) \approx \text{const}|k|^{\delta/2}$ of the dispersion relation. The constant C introduced in Eq. (61) is given by the integral

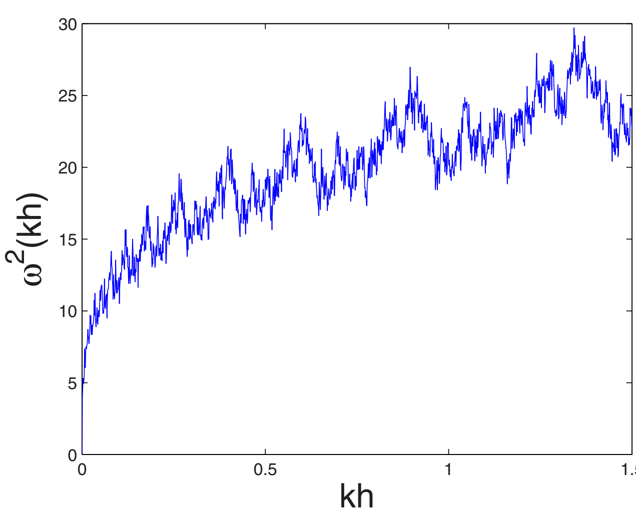

FIG. 3. (Color online) Dispersion relation $\omega^2(kh)$ in arbitrary units for $\delta=0.25$ of fractal dimension $D=1.75$ and $N=1.5$.

$$C = 2\int_0^\infty \frac{(1-\cos\tau)}{\tau^{1+\delta}} d\tau. \tag{62}$$

which exists in the interval $0<\delta<2$.

This approximation holds for "small" $\epsilon \approx \ln N \neq 0 (0<\epsilon \ll 1)$,[7] which corresponds to the limiting case that $N^s = e^v$ is continuous. In this limiting case we obtain the oscillator density from[8] [11]

$$\rho(\omega) = 2\frac{1}{2\pi}\frac{d|k|}{d\omega}, \tag{63}$$

which is normalized such that $\rho(\omega)d\omega$ counts the number (per unit length) of normal oscillators having frequencies within the interval $[\omega, \omega+d\omega]$. We obtain then

$$\rho(\omega) = \frac{2}{\pi\delta h}\left(\frac{\epsilon}{C}\right)^{1/\delta} \omega^{(2/\delta)-1}. \tag{64}$$

where δ is restricted within the interval $0<\delta<2$. We observe hence that the power $2/\delta-1$ is always positive, especially with vanishing oscillator density at $\omega=0$.

We emphasize that neither the dependence on k of Weierstrass-Mandelbrot function (57) is represented by a *continuous* $|k|^\delta$ dependence nor is this function differentiable with respect to k. Application of Eq. (63) is hence only justified to be applied to approximative representation (61) if $0<\epsilon \ll 1$; thus $N=1+\epsilon$ is sufficiently close to one so that N^s is a quasicontinuous function when s runs through $s\in \mathbb{Z}$. Hence Eq. (63) is not generally applicable to Eq. (57) for any arbitrary $N>1$. We can consider Eq. (64) as the low-frequency regime $\omega\to 0$ of the oscillator density holding *only* in the quasicontinuous case $N=1+\epsilon$ with $0<\epsilon\ll 1$.

IV. CONCLUSIONS

We have depicted how self-similar functions and linear operators can be constructed in a simple manner by utilizing a certain category of conventional admissible functions. This approach enables us to construct nonlocal self-similar analogs to the Laplacian and D'Alembertian wave operators. The linear self-similar equation of motion describes the propagation of waves in a quasicontinuous linear chain with harmonic nonlocal self-similar particle interactions. The complexity that comes into play by the self-similarity of the nonlocal interactions is completely captured by the dispersion relations that assume the forms of Weierstrass-Mandelbrot functions (57), exhibiting exact self-similarity and, for certain parameter combinations [relation (60)], fractal features. In a continuum approximation the self-similar Laplacian is expressed in terms of fractional integrals [Eq. (41)] leading for small k (long-wave limit) to a power-law dispersion relation [Eq. (61)] and to a power-law oscillator density [Eq. (64)] in the low-frequency regime.

Self-similar wave operator (53) with the Laplacian (30) can be generalized to describe wave propagation in fractal

[7] $\epsilon=0$ has to be excluded since it corresponds to $N=1$.

[8] The additional prefactor "2" takes into account the two branches of dispersion relation (57) (one for $k<0$ and one for $k>0$).

and self-similar structures that are fractal subspaces embedded in Euclidean spaces of one to three dimensions. The development of such an approach could be a crucial step towards a better understanding of the dynamics in materials with scale hierarchies of internal structures ("multiscale materials"), which may be idealized as fractal and self-similar materials.

We hope to inspire further work and collaborations in this direction to develop appropriate approaches useful for the modeling of static and dynamic problems in self-similar and fractal structures in a wider interdisciplinary context.

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